Value-Peaks of Permutations

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Abstract

In this paper, we focus on a "local property" of permutations: value-peak. A permutation σ has a value-peak $\sigma(i)$ if $\sigma(i-1) < \sigma(i) > \sigma(i+1)$ for some $i \in [2, n-1]$. Define $VP(\sigma)$ as the set of value-peaks of the permutation σ. For any $S \subseteq [3, n]$, define $VP_n(S)$ such that $VP(\sigma) = S$. Let $\mathcal{P}_n = \{S \mid VP_n(S) \neq \emptyset\}$. we make the set $\mathcal{P}_n$ into a poset $\mathscr{P}_n$ by defining $S \preceq T$ if $S \subseteq T$ as sets. We prove that the poset $\mathscr{P}_n$ is a simplicial complex on the set $[3, n]$ and study some of its properties. We give enumerative formulae of permutations in the set $VP_n(S)$.

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1 Introduction

Let $[m,n] := \{m, m+1, \cdots, n\}$. If $m > n$, then $[m,n] = \emptyset$. Let $[n] := [1,n]$ and $\mathfrak{S}_n$ be the set of all the permutations on the set $[n]$. We write permutations of $\mathfrak{S}_n$ in the form $\sigma = (\sigma(1)\sigma(2)\cdots\sigma(n))$. Fix a permutation σ in $\mathfrak{S}_n$. For every $i \in [n-1]$, if $\sigma(i) > \sigma(i+1)$, then we say that i is a position-descent of σ. Define the position-descent set of a permutation σ, denoted by $PD(\sigma)$, as $PD(\sigma) = \{i \in [n-1] \mid \sigma(i) > \sigma(i+1)\}$. Given a set $S \subseteq [n-1]$, suppose $PD(\sigma) = S$ for some $\sigma \in \mathfrak{S}_n$. We easily obtain the increasing and decreasing intervals of σ from the set S. The permutation σ is a function from the set $[n]$ to itself. Since the monotonic property of a function is a global property of the function, the position-descent set of a permutation gives a "global property" of the permutation. We say a permutation $\sigma \in \mathfrak{S}_n$ has a value-descent $\sigma(i)$ if $\sigma(i) > \sigma(i+1)$ for some $i \in [n-1]$. Define the value-descent set of a permutation σ, denoted by $VD(\sigma)$, as $VD(\sigma) = \{\sigma(i) \mid \sigma(i) > \sigma(i+1)\}$. The value-descent set of a permutation is different from its position-descent set. Let $S \subseteq [2,n]$. Suppose $VD(\sigma) = S$ for some $\sigma \in \mathfrak{S}_n$. We only have that k is larger than its immediate right neighbour in the permutation σ for any $k \in S$ and do not obtain the increasing and decreasing intervals of σ from the set S. So the value-descent set of a permutation gives a "local property" of the permutation. For any $S \subseteq [2,n]$, define a set $VD_n(S)$ as $VD_n(S) = \{\sigma \in \mathfrak{S}_n \mid VD(\sigma) = S\}$ and use $vd_n(S)$ to denote the number of permutations in the set $VD_n(S)$, i.e., $vd_n(S) = |VD_n(S)|$. In a joint work [1], Chang, Ma and Yeh derive an explicit formula for $vd_n(S)$.

In this paper, we are interested in another "local property" of permutations: value-peak. A permutation σ has a *value-peak* $\sigma(i)$ if $\sigma(i-1) < \sigma(i) > \sigma(i+1)$ for some $i \in [2, n-1]$. Define $VP(\sigma)$ as the set of value-peaks of σ, i.e., $VP(\sigma) = \{\sigma(i) \mid \sigma(i-1) < \sigma(i) > \sigma(i+1)\}$. For example, the value-peak set of $\sigma = (48362517)$ is $\{5,6,8\}$. Since σ has no value-peaks when $n \leqslant 2$, we may always suppose that $n \geqslant 3$. For any $S \subseteq [n]$, define a set $VP_n(S)$ as $VP_n(S) = \{\sigma \in \mathfrak{S}_n \mid VP(\sigma) = S\}$. Obviously, if $\{1,2,\} \cap S \neq \emptyset$ then $VP_n(S) = \emptyset$.

Example 1.1

$$VP_5(\{4,5\}) = \{ \quad 14253, 14352, 24153, 34152, 24351, 34251, \\ 15243, 15342, 25143, 35142, 25341, 35241 \quad \}.$$

Suppose $S = \{i_1, i_2, \cdots, i_k\}$, where $i_1 < i_2 < \cdots < i_k$. We prove the necessary and sufficient conditions for $VP_n(S) \neq \emptyset$ are $i_j \geqslant 2j+1$ for all $j \in [k]$. Let $\mathcal{P}_n = \{S \mid VP_n(S) \neq \emptyset\}$. We make the set $\mathcal{P}_n$ into a poset $\mathscr{P}_n$ by defining $S \preceq T$ if $S \subseteq T$ as sets. Fig. 1 shows the Hasse diagrams of $\mathscr{P}_3$, $\mathscr{P}_4$ and $\mathscr{P}_5$

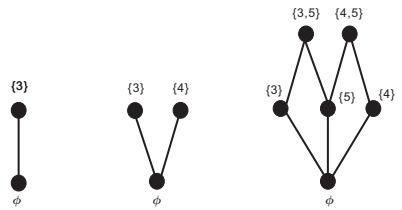

Fig.1. the Hasse diagrams of $\mathscr{P}_3$, $\mathscr{P}_4$ and $\mathscr{P}_5$.

In the next section we prove that $\mathscr{P}_n$ is a simplicial complex on the vertex set $[3,n]$ and derive some properties of $\mathscr{P}_n$.

Then we turn to enumerative problems for permutations by value-peak set. Let $vp_n(S)$ denote the number of permutations in the set $VP_n(S)$, i.e., $vp_n(S) = |VP_n(S)|$. For the cases with $|S| = 0, 1, 2$, we derive explicit formulae for $vp_n(S)$. For general $n \geqslant 3$, we derive the following recurrence relation. Let $n \geqslant 3$ and $S \subseteq [3,n]$. Suppose $VP_n(S) \neq \emptyset$ and let $r = \max S$ if $S \neq \emptyset$, 1 otherwise. For any $0 \leqslant k \leqslant n-r-1$, we have

$$vp_n(S \cup [n-k+1,n]) = 2(k+1)vp_{n-1}(S \cup [n-k,n-1]) + k(k+1)vp_{n-2}(S \cup [n-k,n-2]).$$

For any $S \subseteq [3,n]$, we write the set S in the form $S = \bigcup_{i=1}^{m} [r_i - k_i + 1, r_i]$ such that $r_i \leqslant r_{i+1} - k_{i+1} - 1$ for all $i \in [m-1]$. For example, let $n = 12$ and $S = \{3,4,8,10,11,12\}$. Then $S = [3,4] \cup [8,8] \cup [10,12]$. We have $r_1 = 4$, $k_1 = 2$, $r_2 = 8$, $k_2 = 1$, $r_3 = 12$, $k_3 = 3$. Define the *type* of the set S, denoted $type(S)$, as $(r_1^{k_1}, r_2^{k_2}, \ldots, r_m^{k_m})$. We conclude with a formula for the number of permutations in terms of the type of S.

The paper is organised as follows. In Section 2, we give the necessary and sufficient conditions for $VP_n(S) \neq \emptyset$. We prove the poset $\mathscr{P}_n$ is a simplicial complex on the set $[3,n]$ and study its some properties. In Section 3, we investigate enumerative problems of permutations in the sets $VP_n(S)$. In the Appendix, we list $vp_n(S)$ for $1 \leqslant n \leqslant 8$ obtained by computer searches.

2 The Simplicial Complex $\mathscr{P}_n$

In this section, we give the necessary and sufficient conditions for $VP_n(S) \neq \emptyset$ for any $n \geqslant 3$ and $S \subseteq [n]$. We show $\mathscr{P}_n$ is a simplicial complex on the set $[3,n]$ and study some properties of $\mathscr{P}_n$.

Theorem 2.1 *Let $n \geqslant 3$. Suppose $S = \{i_1, i_2, \cdots, i_k\}$ is a subset of $[n]$, where $i_1 < i_2 < \cdots < i_k$. Then the necessary and sufficient conditions for $VP_n(S) \neq \emptyset$ are $i_j \geqslant 2j+1$ for all $j \in [k]$.*

Proof. Suppose $VP_n(S) \neq \emptyset$ and let $\sigma \in VP_n(S)$. For any $j \in [k]$, all the integers $i_1, i_2, \cdots, i_j$ are a value-peak of σ. Then $i_j - j \geqslant j+1$, hence, $i_j \geqslant 2j+1$.

Conversely, suppose $i_j \geqslant 2j+1$ for all $j \in [k]$. Suppose $[n] \setminus S = \{a_1, a_2, \cdots, a_{n-k}\}$ with $a_1 < a_2 < \cdots < a_{n-k}$. Let σ be the permutation in $\mathfrak{S}_n$ defined by

$$\begin{cases} \sigma(2j) = i_j & \text{for } 1 \leqslant j \leqslant k, \\ \sigma(2j-1) = a_j & \text{for } 1 \leqslant j \leqslant k+1, \\ \sigma(j) = a_j & \text{for } 2k+2 \leqslant j \leqslant n. \end{cases}$$

Obviously, $VP(\sigma) = S$ and $VP_n(S) \neq \emptyset$. ∎

Corollary 2.1 *Let $n \geqslant 3$ and $S \subseteq [n]$. Suppose $VP_n(S) \neq \emptyset$. We have $|S| \leqslant \lfloor \frac{n-1}{2} \rfloor$.*

Proof. Suppose $S = \{i_1, i_2, \cdots, i_k\}$ with $i_1 < i_2 < \cdots < i_k$. Since $VP_n(S) \neq \emptyset$, Theorem 2.1 tells us that $n \geqslant i_k \geqslant 2k+1$. Hence $k \leqslant \lfloor \frac{n-1}{2} \rfloor$. ∎

Corollary 2.2 *Let $n \geqslant 3$ and $S \subseteq [n]$. Suppose $VP_n(S) \neq \emptyset$. Then for $|S| < \lfloor \frac{n-1}{2} \rfloor$, we have $VP_{n+1}(S \cup \{n+1\}) \neq \emptyset$; for $|S| = \lfloor \frac{n-1}{2} \rfloor$, we have $VP_{n+1}(S \cup \{n+1\}) \neq \emptyset$ if n is even; otherwise, $VP_{n+1}(S \cup \{n+1\}) = \emptyset$.*

Proof. Let $k = |S|$. $k < \lfloor \frac{n-1}{2} \rfloor$ implies $2(k+1)+1 \leqslant 2\lfloor \frac{n-1}{2} \rfloor + 1 < n+1$. So, $VP_{n+1}(S \cup \{n+1\}) \neq \emptyset$ when $|S| < \lfloor \frac{n-1}{2} \rfloor$. For the case with $k = \lfloor \frac{n-1}{2} \rfloor$, we have

$$2(k+1)+1 = \begin{cases} n+1 & \text{if } n \text{ is even,} \\ n+2 & \text{if } n \text{ is odd.} \end{cases}$$

By Theorem 2.1, $VP_{n+1}(S \cup \{n+1\}) \neq \emptyset$ if n is even; otherwise, $VP_{n+1}(S \cup \{n+1\}) = \emptyset$. ∎

Following [3], define a *simplicial complex* Δ on a vertex set V as a collection of subsets of V satisfying:
(1) If $x \in V$, then $\{x\} \in \Delta$, and
(2) if $S \in \Delta$ and $T \subseteq S$, then $T \in \Delta$.

Theorem 2.2 *Let $n \geqslant 3$. Then $\mathscr{P}_n$ is a simplicial complex on the set $[3, n]$.*

Proof. Obviously, $\emptyset \in \mathscr{P}_n$. For any $3 \leqslant x \leqslant n$, Theorem 2.1 implies $\{x\} \in \mathscr{P}_n$. Let T be a subset of $[n]$ such that $VP_n(T) = \emptyset$. Note that $VP_n(S) = \emptyset$ for any $T \subseteq S$. Thus given an $S \in \mathscr{P}_n$, we have $T \in \mathscr{P}_n$ for all $T \subseteq S$. Hence, $\mathscr{P}_n$ is a simplicial complex on the set $[3, n]$. ∎

If P and Q are posets, then the *direct product* of P and Q is the poset $P \times Q$ on the set $\{(x, y) \mid x \in P \text{ and } y \in Q\}$ such that $(x, y) \leqslant (x', y')$ in $P \times Q$ if $x \leqslant x'$ in P and $y \leqslant y'$ in Q. Recall that the poset $\mathbf{n}$ is formed by the set $[n]$ with its usual order. By Corollary 2.2, we obtain a method to construct the poset $\mathscr{P}_{n+1}$ from $\mathscr{P}_n$.

Theorem 2.3 *$\mathscr{P}_{n+1} \cong \mathbf{2} \times \mathscr{P}_n$ if n is even; $\mathscr{P}_{n+1} \cong (\mathbf{2} \times \mathscr{P}_n) \setminus (\{1\} \times \mathcal{P}_{n, \lfloor \frac{n-1}{2} \rfloor - 1})$ if n is odd.*

Now, we derive some properties of the simplicial complex $\mathscr{P}_n$. By Theorem 2.3, it is easy to obtain the Möbius function of the poset $\mathscr{P}_n$.

Corollary 2.3 *Let $\mu_n = \mu_{\mathscr{P}_n}$ be the Möbius function of the poset $\mathscr{P}_n$. Then $\mu_n(S, T) = (-1)^{|T|-|S|}$ for any $S \preceq T$ in $\mathscr{P}_n$.*

Proof. Obviously, $\mu_3(\emptyset, \{3\}) = -1$. By induction for n, we assume $\mu_n(S, T) = (-1)^{|T|-|S|}$ for any $S \preceq T$ in $\mathscr{P}_n$. By Theorem 2.3, it follows that

$$\mu_{n+1}(S, T) = \begin{cases} \mu_n(S \setminus \{n+1\}, T \setminus \{n+1\}) & \text{if } n+1 \in S \cap T, \\ \mu_n(S, T) & \text{if } n+1 \notin S \cup T, \\ -\mu_n(S, T \setminus \{n+1\}) & \text{if } n+1 \notin S \text{ and } n+1 \in T \end{cases}$$

for any $S \prec T$. Simple computations show that $\mu_{n+1}(S,T) = (-1)^{|T|-|S|}$. ∎

For every $S \in \mathscr{P}_n$, we call the element S a *face* of $\mathscr{P}_n$ and the *dimension* of S is defined to be $|S|-1$, denoted $\dim(S)$. In particular, the void set $\emptyset$ is always a face of $\mathscr{P}_n$ of dimension -1, i.e., $\dim(\emptyset) = -1$. Also define the dimension of $\mathscr{P}_n$ by $\dim(\mathscr{P}_n) = \max_{S\in\mathscr{P}_n}(\dim(S))$.

Theorem 2.4 *$dim(\mathscr{P}_n) = \lfloor \frac{n-1}{2} \rfloor - 1$.*

Proof. Taking $S = \{3, 5, \cdots, 2\lfloor \frac{n-1}{2} \rfloor + 1\}$, by Theorem 2.1, we have $S \in \mathscr{P}_n$. From Corollary 2.1 it follows that the dimension of $\mathscr{P}_n$ is $\lfloor \frac{n-1}{2} \rfloor - 1$. ∎

Define $\mathcal{P}_{n,i}$ as the set of all the faces of dimension i in $\mathscr{P}_n$, i.e., $\mathcal{P}_{n,i} = \{S \in \mathcal{P}_n \mid |S| = i+1\}$ for any $-1 \leqslant i \leqslant \lfloor \frac{n-1}{2} \rfloor - 1$. Let $p_{n,i} = |\mathcal{P}_{n,i}|$. The sequence $(p_{n,-1}, p_{n,0}, \ldots, p_{n,\lfloor \frac{1}{2}(n-1)\rfloor - 1})$ is called the *f-vector* of the simplicial complex $\mathscr{P}_n$. Define the *f-polynomial* of $\mathscr{P}_n$ as $\mathscr{P}_n(x) = \sum_{i=0}^{\lfloor \frac{1}{2}(n-1)\rfloor} p_{n,i-1} x^{\lfloor \frac{1}{2}(n-1)\rfloor - i}$.

To study the f-vector of $\mathscr{P}_n$, we introduce the concept of left factors of Dyck path. An n-Dyck path is a lattice path in the first quadrant starting at $(0,0)$ and ending at $(2n,0)$ with only two kinds of steps—rise step: $U = (1,1)$ and fall step: $D = (1,-1)$. We can also consider an n-Dyck path P as a word of $2n$ letters using only U and D. Let $L = w_1 w_2 \cdots w_n$ be a word, where $w_j \in \{U, D\}$ and $n \geqslant 0$. If there is another word R which consists of U and D such that LR forms a Dyck path, then L is called an *n-left factor* of Dyck paths. Let $\mathcal{L}_n$ denote the set of all n-left factors of Dyck paths. For any $i \geqslant 0$, let $\mathcal{L}_{n,i}$ denote the set of all n-left factors of Dyck paths from $(0,0)$ to $(n, n-2i)$. It is well known that $|\mathcal{L}_n|$, the cardinality of $\mathcal{L}_n$, equals the nth central binomial number $b_n = \binom{n}{\lfloor \frac{n}{2} \rfloor}$ and $|\mathcal{L}_{n,i}| = \frac{n-2i+1}{i}\binom{n}{i-1}$ (see Cori and Viennot [2]).

In the following lemma, we give a bijection ϕ from the set $\mathcal{P}_n$ to the set $\mathcal{L}_{n-1}$.

Lemma 2.1 *There is a bijection ϕ between the set $\mathcal{P}_n$ and the set $\mathcal{L}_{n-1}$ for any $n \geqslant 3$. Furthermore, the number of elements in $\mathscr{P}_n$ is $\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}$.*

Proof. For any $S \in \mathcal{P}_n$, we construct a word $\phi(S) = w_1 w_2 \cdots w_{n-1}$ as follows:

$$w_i = \begin{cases} D & \text{if} \quad i+1 \in S \\ U & \text{if} \quad i+1 \notin S \end{cases}$$

for any $i \in [n-1]$. Theorem 2.1 implies $\phi(S)$ is an $(n-1)$-left factor of a Dyck path. Conversely, for any an n-left factor $w_1 w_2 \cdots w_{n-1}$ of a Dyck path, let $S = \{i+1 \mid w_i = D\}$. Then $VP_n(S) \neq \emptyset$. Thus the mapping ϕ is a bijection. Note that the number of $(n-1)$-left factors of Dyck paths is $\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}$. Hence, $|\mathcal{P}_n| = \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}$. ∎

Corollary 2.4 *Let $n \geqslant 3$. There is a bijection between the set $\mathcal{P}_{n,i}$ and the set $\mathcal{L}_{n-1,i+1}$ for any $-1 \leqslant i \leqslant \lfloor \frac{n-1}{2} \rfloor - 1$. Furthermore, we have*

$$p_{n,i} = \begin{cases} 1 & if \quad i = -1, \\ \frac{n-2i-2}{i+1}\binom{n-1}{i} & if \quad 0 \leqslant i \leqslant \lfloor \frac{n-1}{2} \rfloor - 1. \end{cases}$$

Proof. We just consider the case with $i \geqslant 0$. For any $S \in \mathcal{P}_{n,i}$, since $|S| = i+1$, the number of the letter D in the word $\phi(S)$ is $i+1$. Hence, $\phi(S)$ is a left factor of a Dyck path from $(0,0)$ to $(n-1, n-2i-3)$. So, $\phi(S) \in \mathcal{L}_{n-1,i+1}$. Hence, $p_{n,i} = |\mathcal{L}_{n-1,i+1}| = \frac{n-2i-2}{i+1}\binom{n-1}{i}$. ∎

Corollary 2.5 *Let $n \geqslant 3$. The sequence $(p_{n,-1}, p_{n,0}, \ldots, p_{n,\lfloor \frac{1}{2}(n-1)\rfloor - 1})$ satisfies the following recurrence relation: for any even integer n,*

$$p_{n+1,i} = \begin{cases} p_{n,i} & \text{if } \ i = -1, \\ p_{n,i-1} + p_{n,i} & \text{if } \ i = 0, 1, \cdots, \frac{n}{2} - 2, \\ p_{n,i-1} & \text{if } \ i = \frac{n}{2} - 1; \end{cases}$$

for any odd integer n,

$$p_{n+1,i} = \begin{cases} p_{n,i} & \text{if } \ i = -1, \\ p_{n,i-1} + p_{n,i} & \text{if } \ i = 0, 1, \cdots, \frac{n-3}{2}, \end{cases}$$

with initial conditions $(p_{3,-1}, p_{3,0}) = (1,1)$.

Proof. First, we consider the case of an even integer n . It is easy to see $p_{n+1,-1} = p_{n,-1} = 1$.

For any $S \in \mathcal{P}_{n+1,\frac{1}{2}n-1}$, Corollary 2.2 tells us $n+1 \in S$. Note that $S \in \mathcal{P}_{n+1,\frac{1}{2}n-1}$ if and only if $S \setminus \{n+1\} \in \mathcal{P}_{n,\frac{1}{2}n-2}$. Hence, $p_{n+1,\frac{1}{2}n-1} = p_{n,\frac{1}{2}n-2}$.

For every $i \in \{0, 1, \ldots, \frac{1}{2}n-2\}$, it is easy to see $\mathcal{P}_{n,i} \subseteq \mathcal{P}_{n+1,i}$. For any $S \in \mathcal{P}_{n+1,i}$ with $n+1 \in S$, $S \setminus \{n+1\}$ can be viewed as an element of $\mathcal{P}_{n,i-1}$. Conversely, for any $S \in \mathcal{P}_{n,i-1}$, Corollary 2.2 implies $S \cup \{n+1\} \in \mathcal{P}_{n+1,i}$. Hence, $p_{n+1,i} = p_{n,i-1} + p_{n,i}$.

Similarly, we can consider the case of an odd integer n. ∎

Theorem 2.5 *Let $n \geqslant 3$.*

(1) The f-polynomial $\mathscr{P}_n(x)$ of the simplicial complex $\mathscr{P}_n$ satisfies the following recurrence relation:

$$x^{\varepsilon(n)}\mathscr{P}_{n+1}(x) = (1+x)\mathscr{P}_n(x) - \varepsilon(n)\frac{2}{n+1}\binom{n-1}{\frac{n-1}{2}}$$

for any n, where $\varepsilon(n) = 0$ if n is even; $\varepsilon(n) = 1$ otherwise, with initial condition $\mathscr{P}_3(x) = x+1$.

(2) Let $\mathscr{P}(x,y) = \sum_{n\geqslant 3} \mathscr{P}_n(x)y^n$. Then $\mathscr{P}(x,y) = \left[\frac{(1+y+xy)[1+x-C(y^2)]}{x-(x+1)^2y^2} - 1\right] y^2$, where $C(y) = \frac{1-\sqrt{1-4y}}{2y}$.

Proof. (1) Obviously, $\mathscr{P}_3(x) = x+1$. Given an odd integer n, we suppose $n = 2i+1$ with $i \geqslant 1$. Corollary 2.5 implies $x\mathscr{P}_{2i+2}(x) = (1+x)\mathscr{P}_{2i+1}(x) - \frac{1}{(i+1)}\binom{2i}{i}$. Similarly, given an even integer n, we suppose $n = 2i$ with $i \geqslant 2$. By Corollary 2.5, we have $\mathscr{P}_{2i+1}(x) = (1+x)\mathscr{P}_{2i}(x)$.

(2) Let $\mathscr{P}_{odd}(x,y) = \sum\limits_{i\geqslant 1} \mathscr{P}_{2i+1}(x)y^{2i+1}$ and $\mathscr{P}_{even}(x,y) = \sum\limits_{i\geqslant 2} \mathscr{P}_{2i}(x)y^{2i}$. We have $\mathscr{P}_{odd}(x,y) = (x+1)y^3 + (x+1)y\mathscr{P}_{even}(x,y)$ and $\mathscr{P}(x,y) = \mathscr{P}_{odd}(x,y) + \mathscr{P}_{even}(x,y)$. It is easy to check $x\mathscr{P}_{2i+3}(x) = (1+x)^2\mathscr{P}_{2i+1}(x) - \frac{1}{i+1}\binom{2i}{i}(x+1)$. So, $\mathscr{P}_{odd}(x,y)$ satisfies the following equation

$$x\mathscr{P}_{odd}(x,y) = (x+1)^2y^2\mathscr{P}_{odd}(x,y) + (x+1)y^3[1+x-C(y^2)],$$

where $C(y) = \frac{1-\sqrt{1-4y}}{2y}$. Equivalently, $\mathscr{P}_{odd}(x,y) = \dfrac{(x+1)y^3[1+x-C(y^2)]}{x-(x+1)^2y^2}$. Hence

$$\mathscr{P}(x,y) = \left[\frac{(1+y+xy)[1+x-C(y^2)]}{x-(x+1)^2y^2} - 1\right] y^2.$$

∎

Let $\mathscr{H}_n(x) = \mathscr{P}_n(x-1) = \sum\limits_{i=0}^{\lfloor\frac{1}{2}(n-1)\rfloor} h_{n,i}x^{\lfloor\frac{1}{2}(n-1)\rfloor - i}$. The polynomial $\mathscr{H}_n(x)$ and the sequence $(h_{n,0}, h_{n,1}, \cdots, h_{n,\lfloor\frac{1}{2}(n-1)\rfloor})$ are called the *h-polynomial* and the *h-vector* of $\mathscr{P}_n$ respectively.

Corollary 2.6 *Let $n \geqslant 3$.*

(1) The h-polynomial $\mathscr{H}_n(x)$ of the simplicial complex $\mathscr{P}_n$ satisfies the recurrence relation:

$$(x-1)^{\varepsilon(n)}\mathscr{H}_{n+1}(x) = x\mathscr{H}_n(x) - \varepsilon(n)\frac{2}{(n+1)}\binom{n-1}{\frac{n-1}{2}}$$

for any n, where $\varepsilon(n) = 0$ if n is even; $\varepsilon(n) = 1$ otherwise, with initial condition $\mathscr{H}_3(x) = x$.

(2) Let $\mathscr{H}(x,y) = \sum\limits_{n\geqslant 3} \mathscr{H}_n(x)y^n$. We have $\mathscr{H}(x,y) = \left[\dfrac{(1+xy)[x-C(y^2)]}{x-1-x^2y^2} - 1\right] y^2$.

Proof. (1) Since $\mathscr{H}_n(x) = \mathscr{P}_n(x-1)$, by Theorem 2.5, we easily obtain $\mathscr{H}_{n+1}(x) = x\mathscr{H}_n(x)$ if n is even, and $(x-1)\mathscr{H}_{n+1}(x) = x\mathscr{H}_n(x) - \frac{2}{n+1}\binom{n-1}{\frac{n-1}{2}}$ if n is odd, with initial condition $\mathscr{H}_3(x) = x$.

(2) Since $\mathscr{H}(x,y) = \mathcal{P}(x-1,y)$, we have $\mathscr{H}(x,y) = \left[\dfrac{(1+xy)[x-C(y^2)]}{x-1-x^2y^2} - 1\right] y^2$. ∎

Corollary 2.7 *Let the sequence $(h_{n,0}, h_{n,1}, \cdots, h_{n,\lfloor\frac{1}{2}(n-1)\rfloor})$ be the h-vector of $\mathscr{P}_n$. Then $h_{n,i}$ satisfies the following recurrence relation:*

$$h_{n+1,i} = \begin{cases} h_{n,0} & \text{if } \ i = 0, \\ h_{n,i} + \varepsilon(n)h_{n+1,i-1} & \text{if } \ 1 \leqslant i \leqslant \lfloor\frac{n}{2}\rfloor - 1, \\ \varepsilon(n)c_{\lfloor\frac{n}{2}\rfloor} & \text{if } \ i = \lfloor\frac{n}{2}\rfloor, \end{cases}$$

where $c_m = \frac{1}{m+1}\binom{2m}{m}$ *and* $\varepsilon(n) = 0$ *if* n *is even; otherwise,* $\varepsilon(n) = 1$*, with initial conditions* $(h_{3,0}, h_{3,1}) = (1, 0)$*. Equivalently,*

$$h_{n,i} = \frac{\lfloor \frac{n}{2} \rfloor - i}{\lfloor \frac{n}{2} \rfloor + i} \binom{\lfloor \frac{n}{2} \rfloor + i}{\lfloor \frac{n}{2} \rfloor}.$$

Proof. The recurrence relations are obtained by comparing coefficients on both sides of the identity in 2.6 (1). Consider $t_{n,i} = \frac{\lfloor \frac{n}{2} \rfloor - i}{\lfloor \frac{n}{2} \rfloor + i}\binom{\lfloor \frac{n}{2} \rfloor + i}{\lfloor \frac{n}{2} \rfloor}$. Note that $t_{n,i}$ and $h_{n,i}$ satisfy the same recurrence relations and $(t_{3,0}, t_{3,1}) = (1, 0)$. Hence,

$$h_{n,i} = t_{n,i} = \frac{\lfloor \frac{n}{2} \rfloor - i}{\lfloor \frac{n}{2} \rfloor + i} \binom{\lfloor \frac{n}{2} \rfloor + i}{\lfloor \frac{n}{2} \rfloor}.$$

∎

Remark 2.1 *Let* $n \geqslant 3$*. The number of left factors of the Dyck path from* $(0, 0)$ *to* $(\lfloor \frac{n}{2} \rfloor + i - 1, \lfloor \frac{n}{2} \rfloor - i - 1)$ *equals* $\frac{\lfloor \frac{n}{2} \rfloor - i}{\lfloor \frac{n}{2} \rfloor + i}\binom{\lfloor \frac{n}{2} \rfloor + i}{\lfloor \frac{n}{2} \rfloor}$ *for any* $0 \leqslant i \leqslant \lfloor \frac{n-1}{2} \rfloor$*.*

Define the *reduced Euler characteristic* of $\mathscr{P}_n$ by $\tilde{\chi}(\mathscr{P}_n) = \sum_{i=0}^{\lfloor \frac{1}{2}(n-1) \rfloor} (-1)^{i-1} p_{n,i-1}$.

Corollary 2.8 *For any* $n \geqslant 3$*,* $\tilde{\chi}(\mathscr{P}_n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \frac{2(-1)^{\frac{n}{2}}}{n}\binom{n-2}{\frac{1}{2}(n-2)} & \text{if } n \text{ is even.} \end{cases}$

Proof. Clearly, $\mathscr{P}_3(-1) = 0$. Theorem 2.5 tells us

$$\mathscr{P}_{n+1}(-1) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{2}{n+1}\binom{n-1}{\frac{1}{2}(n-1)} & \text{if } n \text{ is odd} \end{cases}$$

for any $n \geqslant 4$. Since $\tilde{\chi}(\mathscr{P}_n) = (-1)^{\lfloor \frac{n-1}{2} \rfloor - 1}\mathscr{P}_n(-1)$, we have

$$\tilde{\chi}(\mathscr{P}_n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \frac{2(-1)^{\frac{n}{2}-2}}{n}\binom{n-2}{\frac{1}{2}(n-2)} & \text{if } n \text{ is even.} \end{cases}$$

∎

Let P be a finite post. Define $Z(P, i)$ to be the number of multichains $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{i-1}$ in P for any $i \geqslant 2$. $Z(P, i)$ is called the *zeta polynomial* of P. We state Proposition 3.11.1a and Proposition 3.14.2 in [3] as the following lemma.

Lemma 2.2 [3] *Suppose* P *is a poset.*

(1) Let d_i *be the number of chains* $x_1 < x_2 < \cdots < x_{i-1}$ *in* P*. Then* $Z(P, i) = \sum_{j \geqslant 2} d_j \binom{i-2}{j-2}$.

(2) If P is simplicial and graded, then $Z(P, x+1)$ is the rank-generating function of P.

Corollary 2.9 *Let $n \geqslant 3$ and $i \geqslant 2$. Then*

(1) $Z(\mathscr{P}_n, i) = (i-1)^{\lfloor \frac{n-1}{2} \rfloor} \mathscr{P}_n(\frac{1}{i-1})$ for any $i \geqslant 2$,

(2) $Z(\mathscr{P}_n, i)$ satisfies the recurrence relations:

$$Z(\mathscr{P}_{n+1}, i) = iZ(\mathscr{P}_n, i) - \varepsilon(n)\frac{2(i-1)^{\frac{1}{2}(n+1)}}{n+1}\binom{n-1}{\frac{1}{2}(n-1)},$$

where $\varepsilon(n) = 0$ if n is even; $\varepsilon(n) = 1$ otherwise, with initial condition $Z(\mathscr{P}_3, i) = i$.

(3) Let $Z(x, y) = \sum\limits_{n \geqslant 3} Z(\mathscr{P}_n, x) y^n$. We have

$$Z(x, y) = \left[\frac{(1+xy)[x-(x-1)C(y^2(x-1))]}{1-x^2y^2} - 1\right] y^2.$$

Proof. (1) Let $\mathscr{P}_n(x)$ be the f-polynomial of $\mathscr{P}_n$. We have the rank-generating function of $\mathscr{P}_n$ is $x^{\lfloor \frac{1}{2}(n-1) \rfloor} \mathscr{P}_n(\frac{1}{x})$. Lemma 2.2(2) implies that $Z(\mathscr{P}_n, i) = (i-1)^{\lfloor \frac{n-1}{2} \rfloor} \mathscr{P}_n(\frac{1}{i-1})$.

(2) The recurrence relations for $Z(\mathscr{P}_n, i)$ follow from Theorem 2.5.

(3) Note that $Z(\mathscr{P}_n, x+1) = x^{\lfloor \frac{n-1}{2} \rfloor} \mathscr{P}_n(\frac{1}{x}) = (\sqrt{x})^{n-2+\varepsilon(n)} \mathscr{P}_n(\frac{1}{x})$. By the proof of Theorem 2.5, we have $\mathscr{P}_{odd}(x, y) = \frac{(x+1)y^3[1+x-C(y^2)]}{x-(x+1)^2y^2}$ and $\mathscr{P}_{even}(x, y) = \frac{\mathscr{P}_{odd}(x,y)-(x+1)y^3}{(x+1)y}$. Then

$$\begin{aligned} Z(x+1, y) &= \sum_{n \geqslant 3} (\sqrt{x})^{n-2+\varepsilon(n)} \mathscr{P}_n(\frac{1}{x}) y^n \\ &= \frac{1}{x}\mathscr{P}_{even}(\frac{1}{x}, y\sqrt{x}) + \frac{1}{\sqrt{x}}\mathscr{P}_{odd}(\frac{1}{x}, y\sqrt{x}) \\ &= \left[\frac{(1+y+xy)[1+x-xC(y^2x)]}{1-(x+1)^2y^2} - 1\right] y^2. \end{aligned}$$

∎

Let $d_{\mathscr{P}_n, i}$ be the number of chains $S_{n,1} \prec S_{n,2} \prec \cdots \prec S_{n,i}$ of $\mathscr{P}_n$.

Theorem 2.6 *For any $i \geqslant 1$,*

$$d_{\mathscr{P}_n, i} = \sum \binom{n}{d_1, d_2, \cdots, d_{i+1}} \frac{2d_{i+1} - n}{n},$$

where the sum is over all $(d_1, \cdots, d_{i+1})$ such that $\sum\limits_{k=1}^{i+1} d_k = n$, $d_1 \geqslant 0$, $d_k \geqslant 1$ for all $2 \leqslant k \leqslant i$ and $d_{i+1} \geqslant n - \lfloor \frac{n-1}{2} \rfloor$.

Proof. Let $i \geqslant 1$ and $S_{n,1} \prec S_{n,2} \prec \cdots \prec S_{n,i}$ be a chain of $\mathscr{P}_n$. Suppose $|S_{n,k}| = j_k$ for any $k \in [i]$. Then $0 \leqslant j_1 < j_2 < \cdots < j_i \leqslant \lfloor \frac{n-1}{2} \rfloor$. There are p_{n,j_i-1} ways to obtain the set $S_{n,i}$. Given $S_{n,k}$ with $k \geqslant 2$, there are $\binom{j_k}{j_{k-1}}$ ways to form the subset $S_{n,k-1} \subseteq S_{n,k}$. Hence,

$$\begin{aligned} d_{\mathscr{P}_n,i} &= \sum_{0=j_0 \leqslant j_1 < j_2 < \cdots < j_i \leqslant \lfloor \frac{n-1}{2} \rfloor} \prod_{k=0}^{i-1} \binom{j_{k+1}}{j_k} p_{n,j_i-1} \\ &= \sum \binom{n}{d_1, d_2, \cdots, d_{i+1}} \frac{2d_{i+1}-n}{n}, \end{aligned}$$

where the sum is over all $(d_1, \cdots, d_{i+1})$ such that $\sum\limits_{k=1}^{i+1} d_k = n$, $d_1 \geqslant 0$, $d_k \geqslant 1$ for all $2 \leqslant k \leqslant i$ and $d_{i+1} \geqslant n - \lfloor \frac{n-1}{2} \rfloor$. ∎

Corollary 2.10 *For any $n \geqslant 3$,*

$$\mathscr{P}_n(x) = \sum_{i=2}^{\lfloor \frac{n-1}{2} \rfloor + 2} \frac{x^{\lfloor \frac{n-1}{2} \rfloor + 2 - i}}{(i-2)!} \prod_{j=1}^{i-2} (1-jx) \sum \binom{n}{d_1, d_2, \cdots, d_i} \frac{2d_i - n}{n}$$

where the second sum is over all $(d_1, \cdots, d_i)$ such that $\sum\limits_{k=1}^{i} d_k = n$, $d_1 \geqslant 0$, $d_k \geqslant 1$ for all $2 \leqslant k \leqslant i-1$ and $d_i \geqslant n - \lfloor \frac{n-1}{2} \rfloor$.

Proof. Lemma 2.2(1) implies $Z(\mathscr{P}_n, i) = \sum\limits_{j=2}^{\lfloor \frac{n-1}{2} \rfloor + 2} d_{\mathscr{P}_n,j-1} \binom{i-2}{j-2}$. By Corollary 2.9, we have

$$\mathscr{P}_n\left(\frac{1}{i-1}\right) = \left(\frac{1}{i-1}\right)^{\lfloor \frac{n-1}{2} \rfloor} \sum_{j=2}^{\lfloor \frac{n-1}{2} \rfloor + 2} d_{\mathscr{P}_n,j-1} \binom{i-2}{j-2}$$

for any $i \geqslant 2$. Note that

$$x^{\lfloor \frac{n-1}{2} \rfloor} \sum_{j=2}^{\lfloor \frac{n-1}{2} \rfloor + 2} d_{\mathscr{P}_n,j-1} \binom{\frac{1}{x}-1}{j-2} = \sum_{j=2}^{\lfloor \frac{n-1}{2} \rfloor + 2} \frac{x^{\lfloor \frac{n-1}{2} \rfloor + 2 - j}}{(j-2)!} \prod_{k=1}^{j-2} (1-kx) d_{\mathscr{P}_n,j-1}$$

is a polynomial. Hence, $\mathscr{P}_n(x) = \sum\limits_{j=2}^{\lfloor \frac{n-1}{2} \rfloor + 2} \frac{x^{\lfloor \frac{n-1}{2} \rfloor + 2 - j}}{(j-2)!} \prod\limits_{k=1}^{j-2} (1-kx) d_{\mathscr{P}_n,j-1}$. ∎

3 Enumerations for Permutations in the Set $VP_n(S)$

In this section, we will consider enumerative problems of permutations in the set $VP_n(S)$. Let $vp_n(S)$ denote the number of permutations in the set $VP_n(S)$, i.e., $vp_n(S) = |VP_n(S)|$. First, we need the following lemma.

Lemma 3.1 *Let $n \geqslant 3$ and $S \subseteq [n]$. Suppose $VP_n(S) \neq \emptyset$. Then*

(1) $vp_{n+1}(S) = 2vp_n(S)$, and

(2) let $m = \max S$. We have $vp_n(S) = 2^{n-m}vp_m(S)$ for any $n \geqslant m$.

Proof. (1) It is easy to see $((n+1)\sigma(1)\cdots\sigma(n)) \in VP_{n+1}(S)$ and $(\sigma(1)\cdots\sigma(n)(n+1)) \in VP_{n+1}(S)$ for any $\sigma = (\sigma(1)\cdots\sigma(n)) \in VP_n(S)$. Conversely, for any $\sigma \in VP_{n+1}(S)$, the position of the integer $n+1$ is 1 or $n+1$, i.e., $\sigma^{-1}(n+1) = 1$ or $n+1$, since $n+1 \notin S$. Hence, $vp_{n+1}(S) = 2vp_n(S)$.

(2) Iterating the identity of Lemma 3.1(1), we obtain $vp_n(S) = 2^{n-m}vp_m(S)$. ∎

For any $\sigma \in \mathfrak{S}_n$, let τ be a subsequence $(\sigma(j_1)\sigma(j_2)\cdots\sigma(j_k))$ of $(\sigma(1)\cdots\sigma(n))$, where $1 \leqslant j_1 < j_2 < \cdots < j_k \leqslant n$. Define $\phi_{\sigma,\tau}$ as an increasing bijection of $\{\sigma(j_i) \mid 1 \leqslant i \leqslant k\}$ onto $[k]$. Let $\phi_\sigma(\tau) = (\phi_{\sigma,\tau}(\sigma(j_1))\phi_{\sigma,\tau}(\sigma(j_2))\cdots\phi_{\sigma,\tau}(\sigma(j_k)))$. For the cases with $|S| = 0, 1, 2$, in the following theorem, we derive the explicit formulae for $vp_n(S)$

Theorem 3.1 *Let $n \geqslant 3$. Then*

(1) $vp_n(\emptyset) = 2^{n-1}$,

(2) $vp_n(\{i\}) = 2^{n-2}(2^{i-2}-1)$ for any $i \in [3,n]$, and

(3) $vp_n(\{i,j\}) = 2^{n-3}(2^{i-2}-1)(2^{j-i-1}-1)+2^{n+j-i-5}\cdot 3(3^{i-2}-2^{i-1}+1)$ for any $i,j \in [3,n]$ and $i < j$.

Proof. (1) For any $\sigma \in \mathfrak{S}_n$, suppose the position of the integer 1 is $i+1$, i.e., $\sigma^{-1}(1) = i+1$. Then $\sigma \in VP_n(\emptyset)$ if and only if σ satisfies $\sigma(1) > \cdots > \sigma(i+1) < \cdots < \sigma(n)$. For each integer $j \neq 1$, the position of j has two possibilities at the left or right of the integer 1. Hence, $vp_n(\emptyset) = 2^{n-1}$.

(2) By Lemma 3.1(2), we first consider the number of permutations in the set $VP_i(\{i\})$, where $i \geqslant 3$. For any $\sigma \in VP_i(\{i\})$, suppose the position of the integer i is $k+1$, i.e., $\sigma^{-1}(i) = k+1$. Then $1 \leqslant k \leqslant i-2$, $\phi_\sigma(\sigma(1)\cdots\sigma(k)) \in VP_k(\emptyset)$ and $\phi_\sigma(\sigma(k+2)\cdots\sigma(i)) \in VP_{i-k-1}(\emptyset)$. There are $\binom{i-1}{k}$ ways to form the set $\{\sigma(1),\cdots,\sigma(k)\}$. So, $vp_i(\{i\}) = \sum_{k=1}^{i-2}\binom{i-1}{k}2^{k-1}2^{i-k-2} = 2^{i-2}(2^{i-2}-1)$. Hence, $vp_n(\{i\}) = 2^{n-2}(2^{i-2}-1)$.

(3) It is easy to see the identity holds for $i = 3$ and $j = 4$. By Lemma 3.1(2), we first consider the number of permutations in the set $VP_j(\{i,j\})$, where $3 \leqslant i < j$. We begin from the case $\sigma \in VP_j(\{i,j\})$ with $\sigma^{-1}(i) < \sigma^{-1}(j)$. Let

$$\begin{aligned} T_1(\sigma) &= \{\sigma(k) \mid \sigma(k) < i \text{ and } k < \sigma^{-1}(i)\}, \\ T_2(\sigma) &= \{\sigma(k) \mid \sigma(k) < i \text{ and } \sigma^{-1}(i) < k < \sigma^{-1}(j)\}, \\ T_3(\sigma) &= \{\sigma(k) \mid \sigma(k) < i \text{ and } k > \sigma^{-1}(j)\}. \end{aligned}$$

Note that $T_k(\sigma) \neq \emptyset$ for $k = 1, 2$ since σ has a value-peak i and $\bigcup_{k=1}^{3} T_k(\sigma) = [i-1]$. Let

$$\begin{aligned} T_4(\sigma) &= \{\sigma(k) \mid i < \sigma(k) < j \text{ and } k < \sigma^{-1}(i)\} \\ T_5(\sigma) &= \{\sigma(k) \mid i < \sigma(k) < j \text{ and } \sigma^{-1}(i) < k < \sigma^{-1}(j)\}. \end{aligned}$$

We discuss the following two subcases.

Subcase 1. $T_3(\sigma) = \emptyset$.

Let $T_6(\sigma) = \{\sigma(k) \mid i < \sigma(k) < j, k > \sigma^{-1}(j)\}$. Then $T_6(\sigma) \neq \emptyset$ since σ must have a value-peak j and $\bigcup_{k=4}^{6} T_k(\sigma) = [i+1, j-1]$. For $k = 1, 2, 6$, the subsequences of σ, that are determined by elements from $T_k(\sigma)$, correspond to a permutation in $VP_{|T_k(\sigma)|}(\emptyset)$. The subsequences of σ, that are determined by elements from $T_4(\sigma)$ and $T_5(\sigma)$, are decreasing and increasing, respectively. So, the number of permutations under this subcase is

$$\sum_{(T_1,T_2)} \binom{i-1}{|T_1|,|T_2|} 2^{|T_1|-1}2^{|T_2|-1} \sum_{(T_4,T_5,T_6)} \binom{j-i-1}{|T_4|,|T_5|,|T_6|} 2^{|T_6|-1} = 2^{j-4}(2^{i-2}-1)(2^{j-i-1}-1),$$

where the first sum is over all pairs (T_1, T_2) such that $T_i \neq \emptyset$ for $i = 1, 2$ and $T_1 \cup T_2 = [i-1]$; the second sum is over all triples (T_4, T_5, T_6) such that $T_6 \neq \emptyset$ and $T_4 \cup T_5 \cup T_6 = [i+1, j-1]$.

Subcase 2. $T_3(\sigma) \neq \emptyset$.

Suppose $\min T_3(\sigma) = s$. Let

$$\begin{aligned} T_6(\sigma) &= \{\sigma(k) \mid i < \sigma(k) < j \text{ and } \sigma^{-1}(j) < k < \sigma^{-1}(s)\}, \\ T_7(\sigma) &= \{\sigma(k) \mid i < \sigma(k) < j \text{ and } k > \sigma^{-1}(s)\}. \end{aligned}$$

Then, for $k = 1, 2, 3$, the subsequences of σ, that are determined by elements from $T_k(\sigma)$, correspond to a permutation in $VP_{|T_k(\sigma)|}(\emptyset)$. The subsequences of σ, that are determined by elements from $T_4(\sigma)$ and $T_6(\sigma)$, are decreasing. The subsequences of σ, that are determined by elements from $T_5(\sigma)$ and $T_7(\sigma)$, are increasing. So, the number of permutations under this subcase is

$$\sum_{(T_1,T_2,T_3)} \binom{i-1}{|T_1|,|T_2|,|T_3|} 2^{|T_1|-1}2^{|T_2|-1}2^{|T_3|-1}4^{j-i-1} = 2^{2j-i-6} \cdot 3(3^{i-2} - 2^{i-1} + 1),$$

where the sum is over all triples (T_1, T_2, T_3) such that $T_i \neq \emptyset$ for $i = 1, 2, 3$ and $T_1 \cup T_2 \cup T_3 = [i-1]$.

Similarly, we may consider the case $\sigma \in VP_j(\{i, j\})$ with $\sigma^{-1}(i) > \sigma^{-1}(j)$. Therefore, $vp_j(\{i, j\}) = 2[2^{j-4}(2^{i-2}-1)(2^{j-i-1}-1) + 2^{2j-i-6} \cdot 3(3^{i-2} - 2^{i-1} + 1)]$. In general, for any $n \geqslant 3$ and $3 \leqslant i < j \leqslant n$,

$$vp_n(\{i, j\}) = 2^{n-3}(2^{i-2}-1)(2^{j-i-1}-1) + 2^{n+j-i-5} \cdot 3(3^{i-2} - 2^{i-1} + 1).$$

∎

In the following lemma, we give a recurrence relation for $vp_n(S)$.

Lemma 3.2 *Let $n \geqslant 3$ and $S \subseteq [n-1]$. Then*

$$vp_n(S \cup \{n\}) = [n - 2 - 2|S|]vp_{n-1}(S) + \sum_{j \notin S, j < n} 2vp_{n-1}(S \cup \{j\}).$$

Proof. Suppose $\sigma \in VP_{n-1}(S)$. We want to form a new permutation $\tau \in VP_n(S \cup \{n\})$ by inserting the integer n into σ. For any $j \in S$, since the integer j is a value-peak in the new permutation, we can not insert n into σ beside j. But the integer n must be a value-peak. So, there are $(n-2-2|S|)$ ways to form a new permutation τ from σ such that $\tau \in VP_n(S \cup \{n\})$.

For any $j \notin S$ with $j < n$ and $\sigma \in VP_{n-1}(S \cup \{j\})$, we must insert n into σ beside j such that n becomes a value-peak. So, there are 2 ways to form a new permutation τ from σ such that $\tau \in VP_n(S \cup \{n\})$.

Hence, $vp_n(S \cup \{n\}) = [n-2-2|S|]vp_{n-1}(S) + \sum_{j \notin S, j<n} 2 \cdot vp_{n-1}(S \cup \{j\})$. ∎

For any $S \in [n]$, suppose $S = \{i_1, i_2, \ldots, i_k\}$. Let $\mathbf{x}_S$ stand for the monomial $x_{i_1}x_{i_2}\cdots x_{i_k}$; In particular, let $\mathbf{x}_\emptyset = 1$. Given $n \geqslant 3$, we define a generating function as follows

$$g_n(x_1, x_2, \ldots, x_n; y) = \sum_{\sigma \in \mathfrak{S}_n} \mathbf{x}_{VP(\sigma)} y^{|VP(\sigma)|}.$$

We also write $g_n(x_1, x_2, \ldots, x_n; y)$ as g_n for short. By the recurrence relation as above, we obtain the following result for the generating function g_n.

Corollary 3.1 *Let $n \geqslant 3$ and $g_n = \sum_{\sigma \in \mathfrak{S}_n} \mathbf{x}_{VP(\sigma)} y^{|VP(\sigma)|}$. Then g_n satisfies the following recursion:*

$$g_{n+1} = [2 + (n-1)x_{n+1}y]g_n + 2x_{n+1}\sum_{i=1}^{n} \frac{\partial g_n}{\partial x_i} - 2x_{n+1}y^2\frac{\partial g_n}{\partial y}.$$

for all $n \geqslant 3$ with initial condition $g_3 = 4 + 2x_3y$, where the notation "$\frac{\partial g_n}{\partial y}$" denotes partial differentiation of g_n with respect to y.

Proof. Obviously, $g_3 = 4 + 2x_3y$ and $\sum_{\sigma \in \mathfrak{S}_n} \mathbf{x}_{VP(\sigma)} y^{|VP(\sigma)|} = \sum_{S \subseteq [2,n]} vp_n(S)\mathbf{x}_S y^{|S|}$. Hence,

$$\begin{aligned}
g_{n+1} &= \sum_{S \subseteq [n+1]} vp_{n+1}(S)\mathbf{x}_S y^{|S|} \\
&= \sum_{S \subseteq [n+1], n+1 \in S} vp_{n+1}(S)\mathbf{x}_S y^{|S|} + \sum_{S \subseteq [n+1], n+1 \notin S} vp_{n+1}(S)\mathbf{x}_S y^{|S|} \\
&= \sum_{S \subseteq [n]} \left[(n-1-2|S|)vp_n(S) + \sum_{i \in [n]\setminus S} 2vp_n(S \cup \{i\}) \right] \mathbf{x}_S x_{n+1} y^{|S|+1} + 2g_n \\
&= 2\sum_{S \subseteq [n]} \sum_{i \in [n]\setminus S} vp_n(S \cup \{i\})\mathbf{x}_S x_{n+1} y^{|S|+1} - 2\sum_{S \subseteq [n]} |S| vp_n(S)\mathbf{x}_S x_{n+1} y^{|S|+1} \\
&\quad + [2 + (n-1)x_{n+1}y]g_n.
\end{aligned}$$

Note that

$$\frac{\partial g_n}{\partial y} = \sum_{S \subseteq [n]} |S| vp_n(S)\mathbf{x}_S y^{|S|-1}$$

and

$$\begin{aligned}
&\sum_{S\subseteq[n]}\sum_{i\in[n]\setminus S} vp_n(S\cup\{i\})\mathbf{x}_S x_{n+1}y^{|S|+1}\\
=\ &\sum_{S\subseteq[n],S\neq\emptyset} vp_n(S)x_{n+1}y^{|S|}\sum_{i\in S}\frac{\mathbf{x}_S}{x_i}\\
=\ &x_{n+1}\sum_{i=1}^{n}\frac{\partial g_n}{\partial x_i}.
\end{aligned}$$

Therefore, $g_{n+1}=[2+(n-1)x_{n+1}y]g_n+2x_{n+1}\sum_{i=1}^{n}\frac{\partial g_n}{\partial x_i}-2x_{n+1}y^2\frac{\partial g_n}{\partial y}$. ∎

By computer search, we obtain $vp_n(S)$ for all $3\leqslant n\leqslant 8$ and $S\subseteq[3,n]$ and list them in Appendix. In Table 1., we give the generating functions g_n for $3\leqslant n\leqslant 5$.

The generating function g_n for $3\leqslant n\leqslant 5$
$g_3=4+2x_3y$
$g_4=8+4x_3y+12x_4y$
$g_5=16+8x_3y+24x_4y+56x_5y+4x_3x_5y^2+12x_4x_5y^2$

Table 1. The generating function g_n for $3\leqslant n\leqslant 5$.

Corollary 3.2 *Let $n\geqslant 3$ and $S\subseteq[3,n]$.*

(1) Suppose $S=\{i_1,\ldots,i_k\}$, where $i_1<i_2<\ldots<i_k$. If there exists $j\in[k]$ such that $i_j=2j+1$, then $vp_n(S\cup\{n\})=[n-2-2|S|]vp_{n-1}(S)+\sum_{i\notin S,2j+2<i<n}2vp_{n-1}(S\cup\{i\})$.

(2) $vp_n(\{3,5,\ldots,2k+1\})=2^{n-k-1}$ for all $k\in[\lfloor\frac{n-1}{2}\rfloor]$.

Proof. (1) By Theorem 2.1, $VP_n(S\cup i)=\emptyset$ for any $i\notin S$ and $i<2j+1$ since $i_j=2j+1$. We immediately obtain the results as desired.

(2) By induction on k. For $\bar{k}=1$, by Theorem 3.1(2), we have $vp_n(\{3\})=2^{n-2}$. Suppose the identity holds for any $\bar{k}=k$. For $\bar{k}=k+1$, by Lemma 3.2 and the induction hypothesis, $vp_{2k+3}(\{3,5,\ldots,2k+3\})=vp_{2k+2}(\{3,5,\ldots,2k+1\})=2vp_{2k+1}(\{3,5,\ldots,2k+1\})=2\cdot 2^k=2^{k+1}$. Hence $vp_n(\{3,5,\ldots,2k+3\})=2^{n-2k-3}\cdot 2^{k+1}=2^{n-k-2}$. ∎

Now, we give another recurrence relation for $vp_n(S)$.

Lemma 3.3 *Let $n\geqslant 3$ and $S\subseteq[3,n]$. Suppose $VP_n(S)\neq\emptyset$ and let $r=\max S$ if $S\neq\emptyset$, 1 otherwise. For any $0\leqslant k\leqslant n-r-1$, we have*

$$vp_n(S\cup[n-k+1,n])=2(k+1)vp_{n-1}(S\cup[n-k,n-1])+k(k+1)vp_{n-2}(S\cup[n-k,n-2]).$$

Proof. For any $\sigma\in VP_n(S\cup[n-k+1,n])$, we consider the following four cases.

Case 1. There are no integers $i\in[n-k+1,n]$ such that the position of i is beside $n-k$ in σ, i.e., $|\sigma^{-1}(i)-\sigma^{-1}(n-k)|=1$. Then $\sigma^{-1}(n-k)=1$ or n since the permutation σ

has not a value-peak $n-k$. We obtain a new permutation τ by exchanging the positions of $n-k$ and n in σ. Clearly, $\tau \in VP_n(S \cup [n-k, n-1])$. Lemma 3.1 (1) tells us $vp_n(S \cup [n-k, n-1]) = 2vp_{n-1}(S \cup [n-k, n-1])$. Hence, the number of permutations under this case is $2 \cdot vp_{n-1}(S \cup [n-k, n-1])$.

Case 2. There are exactly two integers $j, m \in [n-k+1, n]$ such that $|\sigma^{-1}(j) - \sigma^{-1}(n-k)| = 1$ and $|\sigma^{-1}(m) - \sigma^{-1}(n-k)| = 1$. Deleting j and m, we obtain a subsequence τ of σ . Then $\phi_\sigma(\tau) \in VP_{n-2}(S \cup [n-k, n-2])$. Note that there are $k(k-1)$ ways to form the pairs (j, m). Hence, the number of permutations under this case is $k(k-1)vp_{n-2}(S \cup [n-k, n-2])$.

Case 3. There is exactly one integer $j \in [n-k+1, n]$ such that $|\sigma^{-1}(j) - \sigma^{-1}(n-k)| = 1$. Then there are k ways to form the set $\{j\}$. Let τ be the subsequence of σ obtained by deleting j. There are the following two subcases.

Subcase 3.1. $\sigma^{-1}(n-k) \neq 1$ and n. Then $\phi_\sigma(\tau) \in VP_{n-1}(S \cup [n-k, n-1])$. Hence, the number of permutations under this subcase is $k \cdot vp_{n-1}(S \cup [n-k, n-1])$.

Subcase 3.2. $\sigma^{-1}(n-k) = 1$ or n. Then $\phi_\sigma(\tau) \in VP_{n-2}(S \cup [n-k, n-2])$. Hence, the number of permutations under this subcase is $k \cdot vp_{n-2}(S \cup [n-k, n-2])$.

So,

$$\begin{aligned} & vp_n(S \cup [n-k+1, n]) \\ = \; & 2vp_{n-1}(S \cup [n-k, n-1]) + k(k-1)vp_{n-2}(S \cup [n-k, n-2]) \\ & +2k \cdot vp_{n-1}(S \cup [n-k, n-1]) + 2k \cdot vp_{n-2}(S \cup [n-k, n-2]) \\ = \; & 2(k+1)vp_{n-1}(S \cup [n-k, n-1]) + k(k+1)vp_{n-2}(S \cup [n-k, n-2]). \end{aligned}$$

∎

Now we associate the recurrence relation in Lemma 3.3 with a lattice path in the plane $\mathbb{Z} \times \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers. In particular, let (n, k), $(n-1, k)$ and $(n-2, k-1)$ be three vertices in the plane $\mathbb{Z} \times \mathbb{Z}$. We get a step $(1, 0)$ (resp. (2,1)) by connecting the vertex $(n-1, k)$ (resp. $(n-2, k-1)$) to the vertex (n, k) and give this step a weight $2(k+1)$ (resp. $k(k+1)$). Fig. 2 shows the resulting graph.

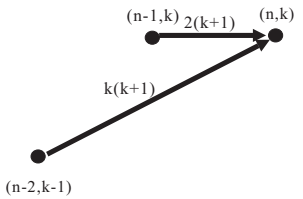

Fig. 2. the graph resulting from the recurrence relation.

Fixing a set S, let the weight of the vertex (n, k) be $vp_n(S \cup [n-k+1, n])$. It is easy to see we can obtain the recurrence relation for $vp_n(S)$ by Fig. 2. So we introduce the concept of value-peak path in the plane $\mathbb{Z} \times \mathbb{Z}$ as follows.

A *value-peak path* is a lattice path in the first quadrant starting at $(0, 0)$ and ending at (n, k) with only two kinds of steps—*horizon step* $H = (1, 0)$ and *rise step* $R = (2, 1)$. We also consider a value-peak path P from $(0, 0)$ to (n, k) as a word of $n-k$ letters using only H and R. Let $P_{n,k}$ be the set of all the value-peak paths from $(0, 0)$ to (n, k). Let

i be a nonegative integer and $P = e_1e_2\cdots e_{n-k} \in P_{n,k}$. For every $j \in [n-k]$, define the weight $w_i(e_j)$ of the step e_j as follows: if the step e_j connects a vertex (x,y) to a vertex $(x+1,y)$, then $w_i(e_j) = 2i+2(y+1)$; if the step e_j connects a vertex (x,y) to a vertex $(x+2,y+1)$, then $w_i(e_j) = (y+i+1)(y+i+2)$. Furthermore, define the weight of the value-peak path P, denoted $w_i(P)$, as $w_i(P) = \prod_{j=1}^{n-k} w_i(e_j)$ and $w(i;n,k) = \sum_{P\in P_{n,k}} w_i(P)$. For any $i < 0$, let $w(i;n,k) = 0$.

Example 3.1 Let $n = 8$, $k = 3$ and $i = 0$. We draw a value-peak path $P = e_1e_2e_3e_4e_5 = HRRHR$ from $(0,0)$ to $(8,3)$ in Fig. 3. For every step e_j in P, we give a label on the step to denote the weight of e_j, i.e., $w_0(e_1) = 2$, $w_0(e_2) = 2$, $w_0(e_3) = 6$, $w_0(e_4) = 6$, $w_0(e_5) = 12$. Hence, $w_0(P) = 1728$.

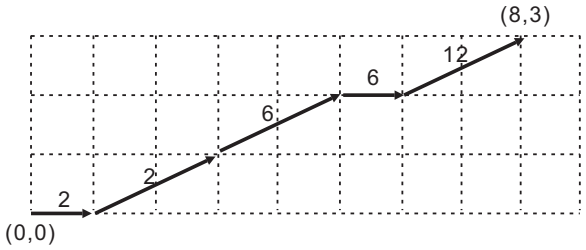

Fig. 3. A value-peak path P with weights from $(0,0)$ to $(8,3)$.

Lemma 3.4 $w(i;n,k) = \dfrac{(i+k)!(i+k+1)!}{i!(i+1)!}[x^{n-2k}]\displaystyle\prod_{m=0}^{k}\frac{1}{1-2(i+1+m)x}$.

Proof. Suppose $P = e_1e_2\cdots e_{n-k} \in P_{n,k}$ and let $\mathcal{R} = \{j \mid e_j = R\}$, then $|\mathcal{R}| = k$. Furthermore, suppose $\mathcal{R} = \{e_{j_1},\cdots,e_{j_k}\}$, where $0 = j_0 < j_1 < j_2 < \cdots < j_k \leqslant n-k-r = j_{k+1}$ and it follows that

$$w_i(P) = \prod_{m=0}^{k}[2i+2m+2]^{j_{m+1}-j_m-1}\prod_{m=0}^{k-1}(m+i+1)(m+i+2).$$

Let $t_m = j_{m+1} - j_m - 1$ for any $0 \leqslant m \leqslant k$. Then $t_m \geqslant 0$ and $\sum_{m=0}^{k} t_m = n-2k$. So,

$$\begin{aligned} w(i;n,k) &= \sum\prod_{m=0}^{k}[2i+2m+2]^{t_m}\prod_{m=0}^{k-1}(m+i+1)(m+i+2) \\ &= \frac{(i+k)!(i+k+1)!}{i!(i+1)!}\sum\prod_{m=0}^{k}[2(i+m+1)]^{t_m} \end{aligned}$$

where the sum is over all $(k+1)$-tuples $(t_0,t_1,\cdots,t_k)$ such that $\sum_{m=0}^{k} t_m = n-r-2k$ and $t_m \geqslant 0$. It is easy to see the sum is the coefficient of x^{n-2k} in the power series $\prod_{m=0}^{k}\frac{1}{1-2(i+1+m)x}$. This completes the proof. ∎

Lemma 3.5 *Let $n \geqslant 3$ and $S \subseteq [3,n]$. Then*

(1) $vp_n([n-k+1,n]) = w(0;n-1,k)$.

(2) Suppose $S \neq \emptyset$ and $VP_n(S) \neq \emptyset$. Let $r = \max S$. For any $0 \leqslant k \leqslant n-r-1$, we have

$$vp_n(S \cup [n-k+1,n]) = \sum_{i=0}^{m} w(k-i;m+i,i)vp_{n-m-i}(S \cup [r+1,n-m-i]),$$

where $m = n-r-k$.

Proof. (1) Fix $k \geqslant 0$. By induction on $n \geqslant k$. For $\bar{n} = k$, we have $vp_k([1,k]) = 0$. It is easy to see $w(0;k-1,k) = k!(k+1)![x^{-k-1}] \prod_{m=0}^{k} \frac{1}{1-2(m+1)x} = 0$. Hence, the identity holds for $\bar{n} = k$. Suppose the identity holds for all $\bar{n} \leqslant n$. For $\bar{n} = n+1$, by Lemma 3.3 and the induction hypothesis,

$$\begin{aligned} & vp_{n+1}([n-k+2,n+1]) \\ = \;& 2(k+1)vp_n([n-k+1,n]) + k(k+1)vp_{n-1}([n-k+1,n-1]) \\ = \;& 2(k+1)w(0;n-1,k) + k(k+1)w(0;n-2,k-1) \\ = \;& w(0;n,k). \end{aligned}$$

Thus the identity holds for $\bar{n} = n+1$.

(2) Let us apply induction on $\bar{m} = n-r-k$. For $\bar{m} = 1$, we have $n-k = r+1$. By Lemma 3.3,

$$\begin{aligned} & vp_n(S \cup [n-k+1,n]) \\ = \;& 2(k+1)vp_{n-1}(S \cup [r+1,n-1]) + k(k+1)vp_{n-2}(S \cup [r+1,n-2]) \\ = \;& w(k;1,0)vp_{n-1}(S \cup [r+1,n-1]) + w(k-1;2,1)vp_{n-2}(S \cup [r+1,n-2]) \\ = \;& \sum_{i=0}^{\bar{m}} w(k-i;\bar{m}+i,i)vp_{n-\bar{m}-i}(S \cup [r+1,n-\bar{m}-i]). \end{aligned}$$

Hence the identity holds for $\bar{m} = 1$. Suppose the identity holds for $\bar{m} = m$. For $\bar{m} = m+1 = n-r-k$, by Lemma 3.3,

$$\begin{aligned} vp_n(S \cup [n-k+1,n]) \;=\; & 2(k+1)vp_{n-1}(S \cup [n-k,n-1]) \\ +\; & k(k+1)vp_{n-2}(S \cup [n-k,n-2]). \end{aligned}$$

By the induction hypothesis,

$$vp_{n-1}(S \cup [n-k,n-1]) \;=\; \sum_{i=0}^{m} w(k-i;m+i,i)vp_{n-1-m-i}(S \cup [r+1,n-1-m-i])$$

and

$$\begin{aligned} & vp_{n-2}(S \cup [n-k, n-2]) \\ = & \sum_{i=0}^{m} w(k-1-i; m+i, i) vp_{n-2-m-i}(S \cup [r+1, n-2-m-i]) \\ = & \sum_{i=1}^{m+1} w(k-i; m-1+i, i-1) vp_{n-1-m-i}(S \cup [r+1, n-1-m-i]). \end{aligned}$$

It is easy to see

$$2(k+1)w(k-i; m+i, i) + k(k+1)w(k-i; m+i-1.i-1) = w(k-i; m+1+i, i)$$

for all $i \in [m]$,

$$2(k+1)w(k; m, 0) = w(k; m+1, 0)$$

and

$$k(k+1)w(k-m-1; 2m, m) = w(k-m-1; 2(m+1), m+1).$$

Hence, $vp_n(S \cup [n-k+1, n]) = \sum_{i=0}^{m+1} w(k-i; m+1+i, i) vp_{n-1-m-i}(S \cup [r+1, n-1-m-i])$. ∎

For any $S \subseteq [3, n]$, recall that type(S) denotes the type of the set S.

Theorem 3.2 *Let $n \geqslant 3$ and $S \subseteq [3, n]$. Suppose type$(S) = (r_1^{k_1}, r_2^{k_2}, \cdots, r_m^{k_m})$ with $m \geqslant 2$ and $VP_n(S) \neq \emptyset$. Let $r_0 = 0$, $A_i = r_i - r_{i-1} - k_i$ and $B_i = \sum_{j=i}^{m} k_j$ for any $1 \leqslant i \leqslant m$. Then*

$$\begin{aligned} vp_n(S) = & 2^{n-r_m} \sum_{i_m=0}^{A_m} \sum_{i_{m-1}=0}^{A_{m-1}} \cdots \sum_{i_2=0}^{A_2} \left[\prod_{s=2}^{m} w(B_s - \sum_{j=s}^{m} i_j; A_s + i_s, i_s) \right. \\ & \left. \cdot w(0; A_1 + B_1 - \sum_{j=2}^{m} i_j - 1, B_1 - \sum_{j=2}^{m} i_j) \right]. \end{aligned}$$

Proof. By induction on m. For $\bar{m} = 2$, by Lemma 3.5,

$$\begin{aligned} vp_{r_2}(S) = & \sum_{i_2=0}^{A_2} w(k_2 - i_2; A_2 + i_2, i_2) vp_{r_1+k_2-i_2}([r_1 - k_1 + 1, r_1 + k_2 - i_2]) \\ = & \sum_{i_2=0}^{A_2} w(B_2 - i_2; A_2 + i_2, i_2) w(0; A_1 + B_1 - i_2 - 1, B_1 - i_2). \end{aligned}$$

Suppose the identity holds for $\bar{m} = m$. For $\bar{m} = m+1$, by Lemma 3.5,

$$vp_{r_{m+1}}(S) = \sum_{t=0}^{A_{m+1}} w(k_{m+1} - t; A_{m+1} + t, t) vp_{r_m+k_{m+1}-t}(S_t),$$

where $type(S_t) = (r_1^{k_1}, r_2^{k_2}, \cdots, (r_m + k_{m+1} - t)^{k_m + k_{m+1} - t})$. For every $0 \leqslant t \leqslant A_{m+1}$, note that $A'_{t,i} = A_i$ and $B'_{t,i} = B_i - t$ for any $1 \leqslant i \leqslant m$. By the induction hypothesis,

$$\begin{aligned} vp_{r_{m+1}}(S) &= \sum_{t=0}^{A_{m+1}} \sum_{i_m=0}^{A_m} \sum_{i_{m-1}=0}^{A_{m-1}} \cdots \sum_{i_2=0}^{A_2} \Bigg[w(0; A_1 + B'_{t,1} - \sum_{j=2}^{m} i_j - 1, B'_{t,1} - \sum_{j=2}^{m} i_j) \\ &\quad \cdot w(k_{m+1} - t; A_{m+1} + t, t) \prod_{s=2}^{m} w(B'_{t,s} - \sum_{j=s}^{m} i_j; A_s + i_s, i_s) \Bigg] \\ &= \sum_{i_{m+1}=0}^{A_{m+1}} \sum_{i_m=0}^{A_m} \sum_{i_{m-1}=0}^{A_{m-1}} \cdots \sum_{i_2=0}^{A_2} \Bigg[\prod_{s=2}^{m+1} w(B_s - \sum_{j=s}^{m+1} i_j; A_s + i_s, i_s) \\ &\quad \cdot w(0; A_1 + B_1 - \sum_{j=2}^{m+1} i_j - 1, B_1 - \sum_{j=2}^{m+1} i_j) \Bigg]. \end{aligned}$$

∎

Example 3.2 Let $n = 8$ and $S = \{3, 7, 8\}$. Then $type(S) = (3^1, 8^2)$, $A_1 = 2$, $A_2 = 3$, $B_1 = 3$ and $B_2 = 2$. By Theorem 3.2, we have

$$\begin{aligned} vp_8(\{3,7,8\}) &= w(2;3,0)w(0;4,3) + w(1;4,1)w(0;3,2) \\ &\quad + w(0;5,2)w(0;2,1) + w(-1;6,3)w(0;1,0). \end{aligned}$$

Note that

$$\begin{aligned} &w(2;3,0) = 216,\ w(0;4,3) = 0,\quad w(1;4,1) = 456,\ w(0;3,2) = 0, \\ &w(0;5,2) = 144,\ w(0;2,1) = 2,\quad w(-1;6,3) = 0,\quad w(0;1,0) = 1. \end{aligned}$$

Thus $vp_8(\{3,7,8\}) = 288$.

4 Appendix

For convenience to check identities given in the previous sections, by computer search, for $1 \leqslant n \leqslant 8$, we obtain the number $vp_n(S)$ of permutations in the set $VP_n(S) \neq \emptyset$ and list them in Table 2.

n									
1	$S=\emptyset$								
	1								
2	$\emptyset$								
	2								
3	$\emptyset$	$\{3\}$							
	4	2							
4	$\emptyset$	$\{3\}$	$\{4\}$						
	8	4	12						
5	$\emptyset$	$\{3\}$	$\{4\}$	$\{5\}$	$\{3,5\}$	$\{4,5\}$			
	16	8	24	56	4	12			
6	$\emptyset$	$\{3\}$	$\{4\}$	$\{5\}$	$\{6\}$	$\{3,5\}$	$\{3,6\}$	$\{4,5\}$	$\{4,6\}$
	32	16	48	112	240	8	24	24	72
	$\{5,6\}$								
	144								
7	$\emptyset$	$\{3\}$	$\{4\}$	$\{5\}$	$\{6\}$	$\{7\}$	$\{3,5\}$	$\{3,6\}$	$\{3,7\}$
	64	32	96	224	480	992	16	48	112
	$\{4,5\}$	$\{4,6\}$	$\{4,7\}$	$\{5,6\}$	$\{5,7\}$	$\{6,7\}$	$\{3,5,7\}$	$\{3,6,7\}$	$\{4,5,7\}$
	48	144	336	288	688	1200	8	24	24
	$\{4,6,7\}$	$\{5,6,7\}$							
	72	144							
8	$\emptyset$	$\{3\}$	$\{4\}$	$\{5\}$	$\{6\}$	$\{7\}$	$\{8\}$	$\{3,5\}$	$\{3,6\}$
	128	64	192	448	960	1984	4032	32	96
	$\{3,7\}$	$\{3,8\}$	$\{4,5\}$	$\{4,6\}$	$\{4,7\}$	$\{4,8\}$	$\{5,6\}$	$\{5,7\}$	$\{5,8\}$
	224	480	96	288	672	1440	576	1376	2976
	$\{6,7\}$	$\{6,8\}$	$\{7,8\}$	$\{3,5,7\}$	$\{3,5,8\}$	$\{3,6,7\}$	$\{3,6,8\}$	$\{3,7,8\}$	$\{4,5,7\}$
	2400	5280	8640	16	48	48	144	288	48
	$\{4,5,8\}$	$\{4,6,7\}$	$\{4,6,8\}$	$\{4,7,8\}$	$\{5,6,7\}$	$\{5,6,8\}$	$\{5,7,8\}$	$\{6,7,8\}$	
	144	144	432	864	288	864	1728	2880	

Table 2. $vp_n(S)$ for $1 \leqslant n \leqslant 8$ with $VP_n(S) \neq \emptyset$.

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